

Chern conjecture and isoparametric hypersurfaces^{*}

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Abstract

In this note we will review the most important results and questions related to Chern conjecture and isoparametric hypersurfaces, as well as their interactions and applications to various aspects in mathematics.

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1 Introduction

In this note we will review the most important results and questions related to Chern conjecture and isoparametric hypersurfaces, as well as their interactions and applications to various aspects in mathematics.

We will start with a brief history of Chern conjecture and its generalizations in Section 2, then we introduce isoparametric hypersurfaces in real space forms, complex space forms, projective spaces, symmetric spaces, general Riemannian manifolds and in particular, exotic spheres in Section 3, and finally in Section 4, we discuss the interactions and applications of Chern conjecture and isoparametric hypersurfaces.

The exposition here is not rather complete, since our emphasis is on open problems and possible future research interests instead of historical results. For a more detailed history of results on the “hypersurface” part of Chern conjecture and

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for an excellent survey on isoparametric hypersurfaces and their generalizations, one can see [53] and [65] (or [13] for an updated one), respectively.

2 Chern conjecture

More than 40 years ago, S.S. Chern, the leader in modern differential geometry, proposed the following problem in several places (cf. [19],[20],[21]).

Problem 2.1. *Consider closed minimal submanifolds M^n immersed in the unit sphere S^{n+m} with second fundamental form of constant length whose square is denoted by σ . Is the set of values for σ discrete? What is the infimum of these values of $\sigma > \frac{n}{2-\frac{1}{m}}$?*

The affirmative hand of the first question, *i.e.*, the set of values for σ should be discrete, is usually called Chern conjecture. Up to now it is still far from a complete solution of this problem even in the case when M is a hypersurface (cf. Problem 105 in [71]). Moreover, as the advances of studies of isoparametric hypersurfaces, people turned the hypersurface case into the following new formulation (cf. [66], [53]).

Conjecture 2.2. *Let M^n be a closed, minimally immersed hypersurface of the unit sphere S^{n+1} with constant scalar curvature. Then M is isoparametric.*

This formulation bases on the fact that all known examples of minimal hypersurfaces with constant scalar curvature (which, by minimality, is equivalent to the condition that the second fundamental form has constant length) in S^{n+1} are isoparametric, *i.e.*, all of their principal curvatures are constant, and σ for these isoparametric hypersurfaces could only attain finite number of values (see Sections below).

Chern conjecture originated from the famous Simons pinching theorem ([56]):

Theorem 2.3. *Let M^n be a closed, minimally immersed submanifold in the unit sphere S^{n+m} and σ the squared norm of its second fundamental form. Then*

$$\int_M \sigma \left(\sigma - \frac{n}{2-\frac{1}{m}} \right) dM \geq 0.$$

In particular, for $\sigma \leq \frac{n}{2-\frac{1}{m}}$ one has either $\sigma = 0$ or $\sigma = \frac{n}{2-\frac{1}{m}}$ identically on M .

This theorem tells that if σ is constant, it can not take any value in the open interval $(0, \frac{n}{2-\frac{1}{m}})$. By his always elegant presentation of calculations with moving frame method, Chern [19] recovered the inequality of Simons above and further he (joint with do Carmo and Kobayashi, cf. [21], see also [38]) classified the submanifolds on which $\sigma \equiv \frac{n}{2-\frac{1}{m}}$ to be only the Clifford minimal hypersurfaces and the Veronese surface (in S^4). Based on these, he proposed Problem 2.1. After that, to this problem of submanifold part, *i.e.*, for submanifolds of high codimension, many contributions had been made and could be concluded in the following theorem (cf. [70], [55], [18], *etc.*).

Theorem 2.4. *Let M^n be a closed, minimally immersed submanifold in the unit sphere S^{n+m} , $m \geq 2$, $\sigma \leq \frac{2n}{3}$ everywhere on M . Then M is either a totally geodesic submanifold or a Veronese surface in S^{2+m} .*

On the other hand, to the hypersurface part of Chern conjecture, Peng and Terng [50] made the first remarkable breakthrough¹:

Theorem 2.5. *Let M^n be a closed, minimally immersed hypersurface in the unit sphere S^{n+1} with constant scalar curvature. Then there exists a positive number $C(n) \geq \frac{1}{12n}$ such that if $\sigma > n$ then $\sigma > n + C(n)$. Furthermore, if $n = 3$ and $\sigma > 3$, then $\sigma \geq 6$.*

From then on, except for 3-dimensional hypersurfaces there's no more essentially affirmative answer to Chern conjecture though considerable improvements of Theorem 2.5 were made by several geometers. For instance, Yang and Cheng (See, for example, [69]) sharpened the constant $C(n)$ to $C(n) \geq \frac{26}{61}n - \frac{16}{61} > \frac{1}{3}n$ and, under the additional assumption that the sum of cubes of the principal curvatures f_3 is constant, $C(n) \geq \frac{13}{15}n - \frac{4}{5} \geq \frac{2}{3}n$. Following [51], Chang [17] showed that if the hypersurface has exactly three pairwise distinct principal curvatures in every point, then it is isoparametric. As for the 3-dimensional hypersurface case, combining results of Almeida-Brito [3] and Chang [15], Conjecture 2.2 was proved to be right even in a more general category:

Theorem 2.6. *Let M^3 be a closed hypersurface immersed in the unit sphere S^4 with constant mean curvature and constant scalar curvature. Then M is isoparametric.*

Furthermore, the classification of 3-dimensional hypersurfaces in S^4 with two constant mean curvature functions² was also established and only one case (vanishing mean curvature and Gauss-Kronecker curvature) there admits non-isoparametric examples (cf. [2],[3],[4],[15],[41],[52]). See [53] for an conclusion of these results, where it mentioned that Bryant conjectured there should be a local version of Theorem 2.6, *i.e.*, minimal hypersurfaces in S^4 with constant scalar curvature are isoparametric, and Chang [16] proved some partial results towards this problem.

As the study of relations of intrinsic invariants and extrinsic invariants goes deeper, it turns out that there're many more Simons-type inequalities for submanifolds in space forms even in the local sense which would certainly get powerful in attacking global problems like Chern conjecture. In 1999, De Smet, Dillen, Verstraelen and Vrancken [24] obtained an inequality involving the (normalized) scalar curvature ρ , normal scalar curvature ρ^\perp , and norm of mean curvature vector field H for submanifolds of codimension 2 in space forms $N(c)$ of constant sectional curvature c , namely

$$\rho + \rho^\perp \leq |H|^2 + c. \quad (1)$$

¹Added in proof: Very recently Ding and Xin [25] sharpened this result by neglecting the condition of constant scalar curvature.

²There're totally 3 mean curvature functions as elementary symmetric polynomials of principal curvatures.

Then they proposed the conjecture that the inequality (1) holds for submanifolds of arbitrary codimension in space forms (the so-called DDVV conjecture, and the inequality (1) is now called the DDVV inequality). Recently, the DDVV conjecture was completely solved by Lu [40] and the authors [30] (where we obtained further the pointwise equality condition) independently with rather different methods (See [31] for a survey). In particular, by algebraic inequalities consisting commutators of shape operators (in matrix form) that derived in the proof of the DDVV inequality (1), Lu [40] generalized the results of [21] and Theorem 2.4 to the following pinching theorem.

Theorem 2.7. *Let M^n be a closed, minimally immersed submanifold in the unit sphere S^{n+m} . Let λ_2 be the second largest eigenvalue of the semi-positive symmetric matrix $S := (\langle A^\alpha, A^\beta \rangle)$ where A^α s ($\alpha = 1, \dots, m$) are the shape operators of M with respect to a given (local) normal orthonormal frame. Suppose $0 \leq \sigma + \lambda_2 \leq n$. Then M is totally geodesic, or is a Clifford minimal hypersurface in $S^{n+1} \subset S^{n+m}$, or is a Veronese surface in $S^4 \subset S^{2+m}$.*

Obviously, $\lambda_2 \leq \frac{1}{2}\sigma$ and thus Theorem 2.7 is an elegant combination of those known pinching results for hypersurfaces and submanifolds of high codimension. Based on this, Lu [40] proposed a similar conjecture as Chern conjecture by taking $\sigma + \lambda_2$ instead of σ :

Conjecture 2.8. *Let M^n be a closed, minimally immersed submanifold in the unit sphere S^{n+m} with constant $\sigma + \lambda_2$. If $\sigma + \lambda_2 > n$, then there is a constant $\varepsilon(n, m) > 0$ such that $\sigma + \lambda_2 > n + \varepsilon(n, m)$.*

Note that for hypersurface case it reduces to the corresponding part of Chern conjecture and was proved by [50]. A difficulty to attack this conjecture would be the non-smoothness of λ_2 in general.

3 Isoparametric hypersurfaces

The surveys on isoparametric hypersurfaces by Thorbergsson [65] and Cecil [13] are so highly professional and excellent that there's no reason for us to forget them and convince ourself to reproduce a lengthy and boring story. Therefore, we would like to pick out some important aspects to our own interests and refer the details to [65], [13].

A (connected) hypersurface M^n in a space form $N^{n+1}(c)$ of constant sectional curvature c is said to be *isoparametric*, if it has constant principal curvatures. An isoparametric hypersurface in \mathbb{R}^{n+1} can have at most two distinct principal curvatures, and it must be an open subset of a hyperplane, hypersphere or a spherical cylinder $S^k \times \mathbb{R}^{n-k}$. This was first proved for $n = 2$ by Somigliana [57] in 1919 (see also Levi-Civita [39]) and for arbitrary n by Segre [54] in 1938. A similar result holds in \mathbb{H}^{n+1} . These two situations can be derived directly from the Cartan identity which implies the number g of distinct principal curvatures must be 1 or 2 as Élie Cartan did in [9] and [11]. However, as Cartan [9, 10, 11, 12] showed in a series of four papers published in the period 1938–1940, the theory of isoparametric

hypersurfaces in the sphere S^{n+1} is much more beautiful and complicated. So far the classification problem in this situation remains open for merely several cases, although much progress has been made after the remarkable works of Münzner [46] which showed that the number g of distinct principal curvatures must be 1, 2, 3, 4, or 6. In fact, the cases when $g \leq 3$ have been completely classified by Cartan. Besides, Cartan and Münzner found that each isoparametric hypersurface in a sphere determines a so-called isoparametric function f ($= F|_{S^{n+1}}$, where $F : \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ is called the *Cartan polynomial*) that satisfies the so-called Cartan-Münzner equations:

$$|\nabla F|^2 = g^2 |x|^{2g-2}, \quad (2)$$

$$\Delta F = \frac{g^2}{2} (m_2 - m_1) |x|^{g-2}, \quad (3)$$

and conversely the set of level hypersurfaces of an isoparametric function f consists of a family of parallel hypersurfaces with constant mean curvature (which in space forms implies each hypersurface has constant principal curvatures) whose focal sets are just the two critical sets M_{\pm} of the isoparametric function (as preimages of the maximum and minimum) and are submanifolds (called the *focal submanifolds*) of codimension $m_1 + 1$, $m_2 + 1$ in S^{n+1} respectively. In fact, Münzner proved further that each isoparametric hypersurface M_t in the family separates the sphere S^{n+1} into two connected components B_1 and B_2 , such that B_1 is a disk bundle with fibers of dimension $m_1 + 1$ over M_+ , and B_2 is a disk bundle with fibers of dimension $m_2 + 1$ over M_- , *i.e.*, they are tubes around M_{\pm} . Moreover, the multiplicities of the distinct principal curvatures $\kappa_1 > \cdots > \kappa_g$ are alternatively m_1 and m_2 ($m_1 = m_2$ for $g = 3$), and the principal curvatures of the focal submanifolds M_{\pm} in any normal direction are the constants $\{\cot \frac{\pi}{g}, \dots, \cot \frac{(g-1)\pi}{g}\}$ with multiplicities alternatively m_2 and m_1 for M_+ (alternatively m_1 and m_2 for M_-). Consequently, the focal submanifolds are (austere) minimal submanifolds.

All known examples of isoparametric hypersurfaces with four principal curvatures are of OT-FKM-type (examples with explicit Cartan polynomials constructed by using representations of Clifford algebras in [47] and [29]) with the exception of two homogeneous families, having multiplicities (m_1, m_2) (arranged to be $m_1 \leq m_2$) being $(2, 2)$ and $(4, 5)$. Beginning with Münzner, many mathematicians, including Abresch [1], Grove and Halperin [36], Tang [62] and Fang [27], found restrictions on the multiplicities of the principal curvatures of an isoparametric hypersurface with four or six principal curvatures. This series of results culminated with the paper of Stolz [58], who proved that the multiplicities of an isoparametric hypersurface with four principal curvatures must be the same as those in the known examples of OT-FKM-type or the two homogeneous exceptions. Cecil, Chi and Jensen [14] and independently Immervoll [37] then showed that if the multiplicities (m_1, m_2) of an isoparametric hypersurface $M^n \subset S^{n+1}$ with four principal curvatures satisfy $m_2 \geq 2m_1 - 1$, then M must be of OT-FKM-type. Combining with results of Takagi [60] and Ozeki-Takeuchi [47] that M must be homogeneous if $m_1 = 1$ or 2, their theorem classified isoparametric hypersurfaces with four principal curvatures for all possible pairs of multiplicities except for four cases, the homogeneous pair $(4, 5)$, and the OT-FKM pairs $(3, 4)$, $(6, 9)$

and (7, 8). By employing more commutative algebra than that explored in [14], Chi [22] gave a proof to the case of (3, 4) that it must be one of the OT-FKM-type. Therefore the cases for the multiplicity pairs (4, 5), (7, 8) and (6, 9) remain open now³.

In the case of an isoparametric hypersurface with six principal curvatures, Münzner showed that all of the principal curvatures must have the same multiplicity m , and Abresch [1] showed that m must equal 1 or 2. By the classification of homogeneous isoparametric hypersurfaces due to Takagi and Takahashi [61], there is only one homogeneous family in each case up to congruence. In the case of multiplicity $m = 1$, Dorfmeister and Neher [26] showed that an isoparametric hypersurface must be homogeneous, thereby completely classifying that case. It has long been conjectured that the one homogeneous family in the case $g = 6$, $m = 2$, is the only isoparametric family in this case, but this conjecture has resisted proof for a long time. The approach that Miyaoka [43] used in the case $m = 1$ shows promise of possibly leading to a proof of this conjecture, but so far a complete proof has not been published⁴.

While the classic theory of isoparametric hypersurfaces in real space forms processed, people came to study similar objects in complex space forms, *i.e.*, hypersurfaces of constant principal curvatures in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ (cf. [59], [5] and references therein). In particular, Berndt and Díaz-Ramos [6, 7] completely classified the real hypersurfaces with 3 distinct constant principal curvatures in $\mathbb{C}H^n$ as that they are all homogeneous. Another generalization is to consider global structures, *i.e.*, a family of parallel hypersurfaces of constant mean curvature, in projective spaces $\mathbb{F}P^n$ ($\mathbb{F} = \mathbb{C}, \mathbb{H}$) just as [67] and [48] did, where [67] showed that there is a correspondence between such “isoparametric” hypersurfaces in $\mathbb{C}P^n$ and isoparametric hypersurfaces in S^{2n+1} by the Hopf fibration, and there are some “isoparametric” hypersurfaces in $\mathbb{C}P^n$ with non-constant principal curvatures, while [48] investigated systematically the number of distinct principal curvatures and their multiplicities of such “isoparametric” hypersurfaces in $\mathbb{F}P^n$. Later, Terng and Thorbergsson [64] introduced a class of submanifolds in simply connected symmetric spaces of compact type that they called equifocal and similar structural results as the classic case were established. A thorough study of the possible values of the multiplicities (m_1, m_2) for equifocal hypersurfaces was done by Tang in [63]. It turns out that equifocal hypersurfaces are the same as the level hypersurfaces of transnormal functions defined in [68] as following.

Definition 3.1. *A non-constant smooth function $f : N \rightarrow \mathbb{R}$ defined on a Riemannian manifold N is called transnormal if there is a smooth⁵ function $b : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$|\nabla f|^2 = b(f). \quad (4)$$

If moreover there is a continuous function $a : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\triangle f = a(f), \quad (5)$$

³Added in proof: Very recently Chi [23] claimed a solution of the cases (4, 5), (6, 9).

⁴From private communications and several geometric conferences, we learnt that R. Miyaoka [44] had completed a proof of this conjecture.

⁵All results would go through for C^2 category instead of C^∞ .

then f is called isoparametric .

As the roles of Cartan-Münzner equations (2,3) in the classic theory, equation (4) implies that the regular hypersurfaces $M_t := f^{-1}(t)$ (where t is a regular value of f) are parallel and (5) implies that these parallel hypersurfaces have constant mean curvatures. These regular level hypersurfaces $M_t := f^{-1}(t)$ of an isoparametric function f are called *isoparametric hypersurfaces*. The preimage of the maximum (resp. minimum) of an isoparametric (or transnormal) function f is called the *focal variety* of f , denoted by M_+ (resp. M_-).

In these terminologies Wang [68] proved a fundamental structural result that the focal varieties M_{\pm} of a transnormal function are smooth submanifolds and each regular level hypersurface M_t is a tube over either of M_{\pm} , which is also compatible with the classic theory of isoparametric hypersurfaces in space forms. Based on this result, the authors [32] improved the fundamental theory of isoparametric functions on general Riemannian manifolds and studied the existence problem of isoparametric (transnormal) functions on certain Riemannian manifolds, especially on exotic spheres. In particular, we proved the non-existence of a properly transnormal function⁶ on any exotic 4-sphere (if exist). On the other hand, we obtained isoparametric examples in some Brieskorn varieties and also in each Milnor sphere. Moreover, we constructed explicitly a properly transnormal but not isoparametric function on the Gromoll-Meyer sphere with two points as the focal varieties, which also differs much from the classic case due to a result claimed in [68] without proof that regular level hypersurfaces of a (properly) transnormal function on S^{n+1} (or \mathbb{R}^{n+1} , but not for \mathbb{H}^{n+1}) are isoparametric (cf. [45] for a proof).

By using Fermi coordinates, we [33] completely proved the minimality of the focal submanifolds of an isoparametric function on a complete Riemannian manifold (claimed in [68] without proof), and obtained the same properties of principal curvatures of the focal submanifolds of an isoparametric function on a complete Riemannian manifold as the classic case under the additional assumption that each isoparametric hypersurface has constant principal curvatures.

There're still many open problems⁷ in this direction, such as the existence problem of an isoparametric function on the Gromoll-Meyer sphere with two points as its focal varieties, the possibilities of the numbers (g, m_1, m_2) for isoparametric hypersurfaces in exotic spheres, the relationship with the normal holonomy theory, etc.

⁶Where by *proper* we mean that the focal varieties of the transnormal function have codimension not less than 2.

⁷Added in proof: Very recently Ge, Tang and Yan [34] introduced the notion of k -th isoparametric hypersurfaces which give a filtration of isoparametric hypersurfaces by the number of constant higher order mean curvatures. Hence many related questions arise such as existence problem or classification problem of k -th isoparametric hypersurfaces in certain Riemannian manifolds.

4 Interactions and Applications

To conclude the story, in this section we would like to discuss some interactions between Chern conjecture and isoparametric hypersurfaces, and also some applications.

First, by the classification of possible values for (g, m_1, m_2) and the simple structures of the second fundamental forms of them, the minimal isoparametric hypersurfaces⁸ and the focal submanifolds in S^{n+1} are the best examples satisfying the Chern conjecture (resp. Conjecture 2.8) that in this set σ (resp. $\sigma + \lambda_2$) takes only finite number of values since it depends only on the values of (g, m_1, m_2) and n . Conversely, although the studies of Chern conjecture related to many geometric or algebraic inequalities, such as the DDVV inequality and its algebraic version, few applications to the theory of isoparametric hypersurfaces could be found in literature. Further, one could consider deducing some Simons-type inequalities in certain Riemannian manifolds more general than in spheres such that isoparametric hypersurfaces in these manifolds take similar roles as those in spheres.

As the theory of isoparametric hypersurfaces processes extensively and fruitfully, it seems that deeper study of its applications worths people's attentions. For instance, by moving frame method, Peng and Tang [49] obtained the Brouwer degrees of gradient maps (looked as maps from S^{n+1} to itself) of isoparametric functions (Cartan polynomials) on S^{n+1} and applied them to harmonic maps between spheres; Ma and Ohnita [42] studied geometry of compact Lagrangian submanifolds in complex hyperquadrics from the viewpoint of the theory of isoparametric hypersurfaces in spheres, where they determined the Hamiltonian stability of compact minimal Lagrangian submanifolds embedded in complex hyperquadrics which are obtained as Gauss images of isoparametric hypersurfaces in spheres with $g(= 1, 2, 3)$ distinct principal curvatures; Ge and Xie [35] studied properties of the gradient map of an isoparametric function (Cartan polynomial) which help them to deduce the Brouwer degrees of gradient maps immediately and to construct a counterexample to the Brézis question (cf. [8]) on the symmetry for the Ginzburg-Landau system in dimension 6, which gives a partial answer towards the Open problem 2 raised by Farina [28]. Note that the theory had also affections in differential topology as we observed in [32], and it is obviously of great interest that further applications can be found.

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⁸There is a unique minimal isoparametric hypersurface in each family of isoparametric hypersurfaces in S^{n+1} , which also holds in a Riemannian manifold of positive Ricci curvature (cf. [32, 33]).

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